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United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Pythagoras

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Pythagoras of Samos (c. 569–475 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Pythagoras.html for more information.

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Version 1.1, Oct 2019

This solution sheet also contains a commentary at the end of each solution for mentors. Mentors might feed ideas to their students. Alternatively they can ignore the comments!

Generally earlier questions are intended to be easier and later questions more difficult.

1. Write down the units digits of all the square numbers from 1^2 to at least 20^2.

How can you tell that 573 is not a square number?

SOLUTION

Only 0, 1, 4, 5, 6 or 9 can occur in the units place of a square number.

2. What is the 2019th letter in the sequence

$$ABCDEDCBABCDEDCBABCDEDCBAB\ldots$$

where the pattern $ABCDEDCB$ is repeated again and again?

ANSWER C

SOLUTION

The pattern repeats every 8 characters so it is the 2016th character which ends the last complete pattern.

3. A 100-sided polygon has all possible diagonals drawn from one chosen vertex to the others.

How many triangles are formed? What is the sum of the interior angles of the 100-sided polygon?

[You should assume that all the interior angles are less than 180°. This is called a *convex* polygon.]

ANSWER 17 640°

SOLUTION

Any vertex can be joined by diagonals to 97 others, producing 98 triangles.

All the angles in the 98 triangles contribute exactly 180° to the sum of the interior angles, making a total of $97 \times 180°$.

> The total of the interior angles is the same even if the polygon is not convex, though the argument is far more subtle in general. The restriction in the question is designed to help the student.

4. To make 'short' pastry one uses flour to fat (butter, margarine, etc.) in the ratio 2 : 1. To make 'flaky' pastry requires a ratio of 4 : 3.

I have 3 kg of flour and 2 kg of fat and wish to use all the ingredients making some of each type of pastry. How much flaky pastry do I make?

ANSWER 3.5 kg

SOLUTION

This is an ideal problem for an algebraic solution. It is simpler to avoid fractions, so choose x such that you use $4x$ kg of flour and $3x$ kg of fat to make $7x$ kg of 'flaky' pastry (matching the given 4 : 3 ratio). This will leave $3 - 4x$ kg of flour and $2 - 3x$ kg of fat with which to make 'short' pastry.

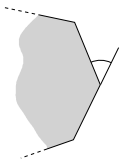
Using the ratio 2 : 1, we now have an equation which we can solve:

$$\begin{aligned} 3 - 4x &= 2(2 - 3x) \\ 3 - 4x &= 4 - 6x && \text{expanding} \\ 6x - 4x &= 4 - 3. && \text{rearranging} \end{aligned}$$

> It is not *necessary* to do this by algebra but it is an ideal technique for this problem. It does *not* require simultaneous equations, though that is an alternative approach.

5. The exterior angles of a triangle are in the ratio 4 : 5 : 6. What is the smallest interior angle?

[An exterior angle of a polygon is the angle between one side and the next side extended, as shown in this sketch.]



ANSWER 36°

SOLUTION

The exterior angles add up to 360°, as they form a whole turn as you walk around the polygon. Thus the exterior angles are 96°, 120° and 144°. The largest exterior angle is adjacent (next to) the smallest interior angle.

6. You tell your friend that there is the same chance of obtaining an even number of heads when tossing four coins as of a head showing when tossing one coin.

Your friend does not believe you.

How can you convince your friend that you are correct?

SOLUTION

An even number of heads means 0, 2 or 4 heads. We imagine tossing the four coins in order, and write down what we get (the *outcome*). For example, the outcome HTHH means that we tossed a head, then tail, then head, then head. This is different from HHTH, where the third toss is a tail.

Two heads appear in the following possible outcomes: HHTT, HTHT, HTTH, THHT, THTH, TTHH. The outcome TTTT has 0 heads, while HHHH has 4 heads, giving a total of eight outcomes with an even number of heads. There are sixteen possible outcomes in total, as each coin can be either heads or tails.

> It is anticipated that a student will just list the cases; this question requires them to be systematic.

7. In a triangle ABC, the length of AB is 2 and the length of AC is 1. C is located so that a circle with BC as diameter passes through A.

If you have not met this idea before, start by drawing the circle and then draw a rectangle inside it with ratio of sides 2 : 1. Draw the diagram carefully and reasonably large. Prove that $BC = \sqrt{5}$.

SOLUTION

$BAC = 90°$ because BC is the diameter of the circle through A.

Thus $AB^2 + AC^2 = BC^2$ leading to $2^2 + 1^2 = BC^2$. Hence $BC^2 = 5$.

> It is quite possible that a student will not know the 90° angle property. Drawing a rectangle inside a circle should convince the student. S/he also needs to be convinced that you cannot inscribe a parallelogram in a circle.

8. Construct a sequence of five different single digit positive integers a, b, c, d, e so that:

b divides into $a + b$ but a does not divide into $a + b$;

c divides into $a + b + c$ but a and b do not do so;

d divides into $a + b + c + d$ but a, b and c do not do so;

e divides into $a + b + c + d + e$ but a, b, c and d do not do so.

ANSWER $a = 8$, $b = 4$, $c = 6$, $d = 3$, $e = 1$

SOLUTION

a cannot be 1. In fact, none of a, b, c or d can be 1. Only e might take the value 1. (Why?)

Possible pairs for a and b to be considered are $4, 2$; $6, 2$; $8, 2$; $6, 3$; $9, 3$ and $8, 4$.

Testing out cases of a, b, c leaves only $4, 2, 3$; $8, 2, 5$; $9, 3, 2$; $9, 3, 4$; $8, 4, 2$; $8, 4, 3$ and $8, 4, 6$ to consider.

Testing out cases for d leaves only $8, 4, 2, 7$; $8, 4, 6, 3$ and $8, 4, 6, 9$.

Then we find the unique solution by trying cases for e: only $8, 4, 6, 3, 1$ works.

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Pythagoras

Sheet 2

Solutions and comments

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Version 1.2, Nov 2020

1. Without using decimal equivalents (or a calculator), prove that these three fractions are arranged in order of size, largest first:

$$\frac{6}{11} \quad \frac{7}{13} \quad \frac{8}{15}$$

Solution

We can check the first two are in order by turning them into the fractions $\frac{78}{143}$ and $\frac{77}{143}$. Similarly writing the second and third fractions as $\frac{105}{195}$ and $\frac{104}{195}$ confirms these are in the correct order.

A second approach is to note that each of these fractions is slightly over $\frac{1}{2}$. If we subtract $\frac{1}{2}$ from each of the fractions, we do not change their order, and we obtain

$$\frac{1}{22} \quad \frac{1}{26} \quad \frac{1}{30}$$

These are clearly in decreasing order of size.

A third, and quite different, approach is to plot the points $A(11, 6)$, $C(13, 7)$ and $B(15, 8)$ on a graph. The gradients of the lines joining $O(0, 0)$ to these three points are the values of the fractions. C is the centre of a parallelogram based on $OAPB$ where P is at $(26, 14)$ (note that as vectors, $(11, 6) + (15, 8) = (26, 14)$), and so the gradient of OC lies between those of OA and OB. The gradient of OA is greater than that of OB since the gradient of AB is $\frac{1}{2}$, which is less than that of OA, and B lies to the right of A on the graph.

We might also note that $\frac{7}{13} = \frac{14}{26}$ is the mediant of $\frac{6}{11}$ and $\frac{8}{15}$, and so lies between them.

2. The sum of two numbers is 80. One is four times the other. What are they?

Answer 16 and 64

Solution

Let the smaller number be x. Then the larger number is $4x$. Hence:

$$5x = 80$$

so $x = 16$ and $4x = 64$.

> This bears similarities to the famous bat-and-ball problem quoted by Daniel Kahneman in *Thinking, Fast and Slow*:
>
> > A bat and ball cost \$1.10.
> > The bat costs one dollar more than the ball.
> > How much does the ball cost?
>
> The 'obvious' answer to our question is 20 and 80.

3. You travel the first 48 miles of a journey at an average speed of 36 mph and the second 48 miles at an average speed of v mph. What is v if you average 45 mph over the whole 96 mile journey?

ANSWER 60 mph

SOLUTION

Using distance = speed × time, we see that it takes 80 minutes for the first 48 miles, and you need to complete the journey in 128 minutes in total in order to average 45 mph over the 96 miles. Hence you have 48 minutes to travel the second 48 miles.

> Note that the answer is not 54 mph simply because 45 is the average of 36 and 54. You are not covering each part of the journey in equal times.

4. Beth has 4 of her friends round for a sleepover. She puts two airbeds down in her own bedroom while a second bedroom has two beds. In how many ways can they split themselves between the two bedrooms if she decides *not* to sleep in her own bed? (Note that this question does not ask in how many ways can they arrange themselves in the beds.)

ANSWER 10

SOLUTION

We can think of this as asking how many ways there are of choosing two of the friends to sleep in the second bedroom. Whether Beth sleeps in her own room or not makes no difference; if she does, she will simply give her bed to one of her friends.

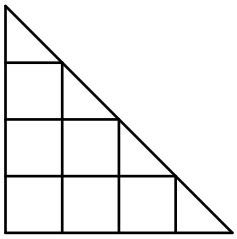
This problem is therefore equivalent to tossing 5 distinct coins and asking how many ways there are of 3 coins showing heads and 2 showing tails.

You may know a way to calculate this efficiently; otherwise, you can just list the possibilities. Such a list might begin as follows, with 1 and 2 identifying the two bedrooms:

Beth	Friend A	Friend B	Friend C	Friend D
1	1	1	2	2
1	1	2	1	2
1	1	2	2	1
1	2	1	1	2

5. How many triangles are there in this figure?

If the diagram were based on half of a 5 by 5 square instead of half of a 4 by 4 square, how many *extra* triangles would there be?

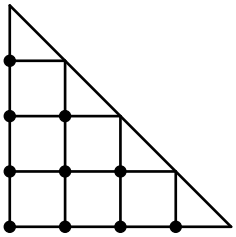


ANSWER 10; 5 extra

SOLUTION

There are 4 triangles with horizontal side 1, 3 with side 2, 2 with side 3 and 1 with side 4. Changing to half a 5 by 5 square increases each of these counts by 1 and also adds the triangle with side 5.

Dots have been added to the figure below to indicate the right angle in each of the triangles; there is only one right-angled triangle for each such point. You might notice from this figure that the number of triangles is a triangular number, and so the general solution $\frac{1}{2}n(n+1)$, where n is the base of the big triangle.



6. The proof of Fermat's Last Theorem was published in 1995. New Year's Day in 1995 was on a Sunday. On what day of the week will New Year's Day be in 2195?

(Try to work it out without asking Google!)

ANSWER Thursday

SOLUTION

If a year is not a leap year, the day of the week on which the following New Year's Day falls is one day later; for example, New Year's Day in 1996 was a Monday. This is because a year has $365 = 52 \times 7 + 1$ days. Following a leap year, New Year's Day is two days later. So every four years, which must include exactly one leap year, New Year's Day advances 5 days round the week.

There are 50 blocks of 4 years in the 200 years from 1995 to 2195. Thus we would expect 1st January to advance 250 days. However, 2100 is *not* a leap year, so 1st January only advances 249 days. As $249 = 35 \times 7 + 4$, New Year's Day in 2195 is 4 days after Sunday, so it falls on a Thursday.

7. When we write 789 in decimal, we mean 7 hundreds plus 8 tens plus 9 ones. We write abc_{10} to mean the number $100a + 10b + c$ (and not a multiplied by b multiplied by c, as in normal algebra).

Show that if $a + b + c$ is divisible by 9, then abc_{10} is also divisible by 9. Show the *converse* too: if abc_{10} is divisible by 9, then so is $a + b + c$.

This gives us a test for determining whether a three-digit number is divisible by 9. Generalise this test to numbers with any number of digits.

SOLUTION

We rewrite

$$100a + 10b + c = (99a + 9b) + a + b + c$$

Since $99a + 9b$ is divisible by 9, it follows that if $a + b + c$ is divisible by 9, so is $100a + 10b + c$.

Similarly, by rewriting this equation as

$$a + b + c = 100a + 10b + c - (99a + 9b)$$

we see that if $abc_{10} = 100a + 10b + c$ is divisible by 9, then so is $a + b + c$.

Generalising this, if we have a number $ab \ldots yz_{10}$, this gives the equation

$$100 \ldots a + 100 \ldots b + \cdots + 10y + z = (99 \ldots a + 99 \ldots b + \cdots + 9y) + a + b + \cdots + y + z;$$

since $99 \ldots a + 99 \ldots b + \cdots + 9y$ is divisible by 9, it follows that if $a + b + \cdots + y + z$ is divisible by 9, so is $ab \ldots yz_{10}$, and the converse is similar.

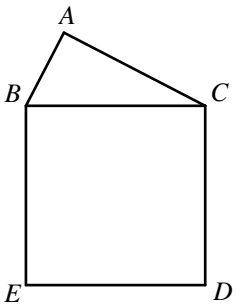
In words, we can say that a number is divisible by 9 if and only if the sum of its digits (when written in decimal) is divisible by 9.

> In fact we can say more: the remainder on dividing abc_{10} by 9 is the same as the remainder on dividing $a + b + c$ by 9. It is not true, though, that the remainder on dividing abc_{10} by 9 is always equal to $a + b + c$; finding some examples of this is instructive.
>
> We could also answer this question using modular arithmetic.

8. In this figure, $\triangle ABC$ is right-angled at A, and $AB < AC$. $BCDE$ is a square.

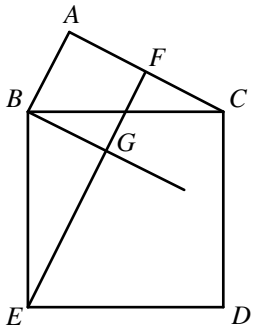
F is on AC so that $\angle AFE = 90°$, G is on EF so that $\angle BGE = 90°$.

Prove that $AFGB$ is a square.



SOLUTION

We begin by sketching in the lines FE and BG:



First we note that by construction, $AFGB$ must be a rectangle, as the angles at A, F and G are all right angles.

Since BG is parallel to AC, then $\angle CBG = \angle ACB$ (by alternate angles).

Now $\angle ABC + \angle CBG = 90°$ and $\angle EBG + \angle CBG = 90°$. Hence $\angle EBG = \angle ABC$.

Both $\triangle ABC$ and $\triangle GBE$ are right-angled, so they are *similar*. Moreover, since $BE = BC$ (as $BCDE$ is a square), these triangles are actually *congruent*.

Thus $BG = AB$, and hence $AFGB$ is a square.

The following is an alternative solution using transformation geometry.

We have noted above that $AFGB$ is a rectangle, and we need to show $BG = AB$.

Rotate $\triangle BAC$ by 90° clockwise about B. (This is the key step that gives this argument a transformation geometry flavour.) Since $BCDE$ is a square, the side BC maps onto BE.

Since $AFGB$ is a rectangle, the point A maps onto the line running along BG. (Note that we *cannot* say that A maps onto G, as we do not yet know that $BA = BG$.)

Since C maps to E, the side AC maps onto the line EF, as EF is perpendicular to AC through E.

Therefore A maps to the intersection of the lines EF and BG, which is G.

Hence $BA = BG$ as required.

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Pythagoras

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Pythagoras of Samos (c. 569–475 BCE).

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Version 1.2, Dec 2020

1. In his will, the first Pharaoh of Ufractia stated: "I bequeath to my oldest child one third of my estate, to the next oldest child one quarter of my estate, and similarly to each succeeding child except the youngest the next unit fraction of my estate. To my youngest I bequeath the remainder of my estate." How many children does he have?

ANSWER (at most) 5 children

SOLUTION

The first two children inherit $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$, leaving $\frac{5}{12}>\frac{1}{5}$, so if there are 4 or more children, the third will inherit $\frac{1}{5}$.

The first three children inherit $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{47}{60}$, leaving $\frac{13}{60}>\frac{1}{6}$, so if there are 5 or more children, the fourth will inherit $\frac{1}{6}$.

The first four children inherit $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{19}{20}$, leaving $\frac{1}{20}<\frac{1}{7}$, so there can be at most one more child, who would inherit the remaining $\frac{1}{20}$ of the estate.

So there are at most 5 children.

Since it is likely from the context of the question that the youngest child should receive the least, there are probably exactly 5 children.

2. In the dark, Emily picks some socks from a drawer which contains 6 black socks, 14 blue socks and 8 green socks. To be sure to pick two of the same colour, what is the minimum number of socks she must pick?

ANSWER 4 socks

SOLUTION

The worst case is that she picks three different colours on the first three picks. The fourth sock is bound to be one of these three colours.

3. Early one Sunday, I was driving along a long road in no particular hurry at a constant 60 mph, with my SatNav telling me that I could expect to arrive at my destination at 11:40 am. After about 8 miles, the SatNav changed the arrival time to 11:41 am. At what speed was the SatNav expecting me to travel on this road to cause this to happen?

ANSWER About 68.5 mph

SOLUTION

At 60 mph, it takes 8 minutes to travel 8 miles. The SatNav expected this distance to be covered in 7 minutes in order to keep to its estimate of an arrival at 11:40 am. On this road, the SatNav was expecting me to travel at $\frac{8}{7} \times 60$ mph.

> The student concerned about the accuracy of the measurements could be more sophisticated about the range of measurements and would come up with a wider range of speed for the final answer, as follows.
>
> At the start of the 8 miles, the estimated arrival time is somewhere between 11:40:00 and 11:40:59. After 8 miles, the estimate arrival time hits 11:41:00. (It cannot be later than this, otherwise the SatNav's prediction would have changed sooner.) So the SatNav expected me to travel these 8 miles between 1 and 60 seconds faster. 8 miles in 7 minutes 59 seconds is a speed of about 60 mph, while 8 miles in 7 minutes is about 68.5 mph. So the expected speed was between about 60 and 68.5 mph.
>
> Even this, though, does not take into account the question's wording of "about 8 miles"; doing so would futher extend the range of possible values. At this point, a better estimate would likely be obtained by just noting the speed limit on the road!

4. Luke writes all the consecutive even numbers from 2 to 2000 (inclusive) on a board.

Louise then erases all the numbers that are multiples of three. How many numbers are left?

ANSWER 667

SOLUTION

Every third even number is a multiple of 3. The first is 6 and the last is 1998, so Louise erases 333 of the 1000 numbers that Luke wrote.

5. How many digits does the answer to 111 222 333 444 555 666 777 888 999 ÷ 37 have?

ANSWER 25

SOLUTION

We note that 37 divides into each triplet of digits 111, 222, 333, etc., as $37 \times 3 = 111$, so the answer is an integer. Thus the quotient is 3 006 009 012 015 .... We see that $111 \div 37 = 3$ is a single digit. For all the other triplets, the quotients are embedded within the long number and so

the quotients are padded out with one or two zeros. There are thus $1 + 8 \times 3 = 25$ digits in the final answer.

6. In the sum TAP + BAT + MAN, each letter represents a different digit, and no number begins with a zero. What is the smallest sum that can be obtained?

ANSWER 610

SOLUTION

T, B and M are the hundreds digits, so must be as small as possible, that is, 1, 2 and 3 in some order. A is in all three tens positions, so has to be as small as possible, so we take A to be 0 (which we have not yet used). We can take P and N to be 4 and 5, and since T also appears as a ones digit, we should take it to be 1. Thus we have $104 + 201 + 305 = 610$.

7. Major Tom has landed on a planet populated by purple cats and black cats. Purple cats always tell the truth and black cats always lie. In the pitch blackness, he meets five cats.

The first cat says: "I'm purple."

The second cat says: "At least three of us are purple."

The third cat says: "The first cat is black."

The fourth cat says: "At least three of us are black."

The fifth cat says: "We're all black."

How many of the five cats are purple?

ANSWER 2 purple cats

SOLUTION

The fifth cat cannot be purple, as purple cats tell the truth. So this cat is lying, and must be black. Therefore there is at least one purple cat.

One of the first and third statements is true and the other is false, so one of these cats is black and the other purple, but it could be either way round.

Thus between the first, third and fifth cats, one is purple and two are black.

The fourth cat cannot be lying, as this would mean that this cat is black and there are at most two black cats, but we already have two other black cats. So the fourth cat must be telling the truth and be purple, which forces the second cat to be black and lying, which is consistent with what the second cat says.

Thus there are three black and two purple cats.

8. How many positive integers n are there such that $8n + 50$ is a multiple of $2n + 1$?

ANSWER 1 solution

SOLUTION

Since $8n + 50 = 4(2n + 1) + 46$, we require $2n + 1$ to divide into 46, whose factors are 1, 2, 23 and 46. Since $2n + 1$ (which is odd and at least 3) cannot equal 1, 2, or 46, the only possibility for $2n + 1$ is 23.

> Note that 46 is the remainder when dividing $8n + 50$ by $2n + 1$.

9. Katrina is standing in a rectangular garden. Her distances from the corners of the garden are 6 m, 7 m, 9 m and d m, where d is an integer. Find d.

ANSWER 2

SOLUTION

d can be 'opposite' the 6 m measurement, the 7 m measurement or the 9 m measurement. Choose the distances x and y to two edges so that $x^2 + y^2 = d^2$. Choose u and v as the distances to the other edges. We can now write four equations for the first case:

$$\begin{aligned} x^2 + y^2 &= d^2 \\ u^2 + v^2 &= 6^2 \\ x^2 + v^2 &= 7^2 \\ y^2 + u^2 &= 9^2 \end{aligned}$$

If we add the top two equations we see that $x^2 + y^2 + u^2 + v^2 = d^2 + 6^2$, but if we add the bottom two equations we get $x^2 + y^2 + u^2 + v^2 = 7^2 + 9^2$. That is, $d^2 + 6^2 = 7^2 + 9^2$. Similarly, the other two cases give $d^2 + 7^2 = 6^2 + 9^2$ and $d^2 + 9^2 = 6^2 + 7^2$. Rearranging these three equations gives the possibilities:

$$\begin{aligned} d^2 &= 7^2 + 9^2 - 6^2 = 94, \\ d^2 &= 9^2 + 6^2 - 7^2 = 68, \\ \text{or} \quad d^2 &= 6^2 + 7^2 - 9^2 = 4. \end{aligned}$$

Only the last possibility gives an integer value for d.

> You might wonder whether it is possible to have a situation with these distances, that is, whether there are values of x, y, u and v for which the four original equations all hold with $d = 2$. And if it is possible, what would those values be?

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Pythagoras

Sheet 4

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Version 1.2, Jan 2021

1. Find the largest value of n such that 9×10^n is a factor of $20!$.

N.B. $n!$ is called the *factorial* of n. It is read as 'factorial n' or 'n factorial'; some people say 'n bang' or 'n shriek' because of the notation. It is the product of all of the whole numbers from 1 to n, so $20! = 1 \times 2 \times 3 \times \cdots \times 19 \times 20$.

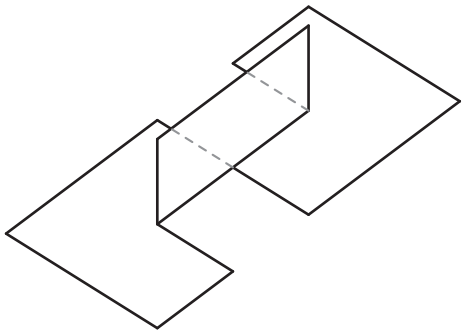
ANSWER 4

SOLUTION

To find the highest power of 10 which is a factor, we look at the prime factors of 10, namely 5 and 2. There are only 4 numbers in the product $1 \times 2 \times \cdots \times 19 \times 20$ that are multiples of 5, namely 5, 10, 15 and 20. None of these are multiples of 5^2, so the total power of 5 in 20! is 4. The power of 2 in 20! is at least 4 (for example, $16 = 2^4$ appears in the product), so the highest power of 10 which is a factor is 10^4.

Now, 9 appears in the product as well, and 9 does not have 5 or 2 as a factor, so 9×10^4 is a factor of 20!.

2. Make the shape shown from a sheet of paper 4 units by 2 units. You are only permitted to cut and fold the paper, but not to glue anything back together. The tab standing up is 2 units by 1 unit in size.



SOLUTION

To change the illustrated figure back to a sheet of paper, imagine you are holding the left end of the shape towards you. Rotate the tab 90° clockwise and the flat part at the right hand end 180° clockwise. You will now see where to make the cuts.

3. In a magic forest there are talking foxes, snakes and turtles. Turtles always speak the truth and foxes always lie. Snakes always lie on rainy days and always tell the truth on fair days. One day, Hogwarts student Hermione Granger talks to four of these animals who tell her one by one:

1st: It's raining today.

2nd: That animal has just lied.

3rd: The weather's fair today.

4th: That animal has just lied or I'm a snake.

At most how many turtles did Hermione talk to?

In the 4th statement, the word "or" means that either the animal has just lied, or I'm a snake, *or both*. This is the standard meaning of the word "or" in mathematics: either or both of the parts are true.

As with many similar logic puzzles, there are several unstated assumptions in this question. For example, we should assume that every day is either rainy or fair, and that every animal is a perfect logician. There is also the unstated instruction, "Based on this information alone, how many . . . ?"

ANSWER 2 turtles at most

SOLUTION

We could lay out two rows of a 'truth table' for the two possibilities of it raining or being fair today.

	Animal 1	Animal 2	Animal 3	Animal 4
Raining	Turtle	Fox or Snake	Fox or Snake	Turtle
Fair	Fox	Turtle or Snake	Turtle or Snake	Fox or Snake

In either case, there are at most two turtles. It is also possible to have no turtles or one turtle.

4. A positive integer is called "good" if it can be written both as a sum of two consecutive integers and as a sum of three consecutive integers. For example, 9 is "good", as $9 = 4 + 5 = 2 + 3 + 4$, but 11 is not good, as $11 = 5 + 6$, but cannot be written as the sum of three consecutive positive integers.

Show that:

(a) 2000, 2002 and 2003 are not "good", but 2001 is "good".

(b) The product of two "good" numbers is "good".

(c) If the product of two positive integers is "good", then at least one of the integers is "good".

SOLUTION

The sum of two consecutive integers, n and $n + 1$ is $2n + 1$. So every "good" number must be odd.

The sum of three consecutive integers, n, $n + 1$ and $n + 2$ is $3n + 3 = 3(n + 1)$. So every "good" number must a multiple of 3.

Putting these together, every "good" number must be an odd multiple of 3.

We must also ask the reverse: is every (positive) odd multiple of 3 a "good" number? If n is odd, then $n = 2m + 1$ for some integer m, so $n = m + (m + 1)$ and is the sum of two consecutive integers. (In the case of $n = 1$, this gives $1 = 0 + 1$; we must allow ourselves to include 0 in the sum for this to work.)

Likewise, if n is a multiple of 3, we can write $n = 3m = (m - 1) + m + (m + 1)$, so n is the sum of three consecutive integers. (Again, in the case of $n = 3$, this gives $3 = 0 + 1 + 2$.)

So "good" numbers are exactly the positive odd multiples of 3.

(a) 2000 and 2002 are even, while 2003 is not a multiple of 3, so they are not "good". But 2001 is an odd multiple of 3, so it is "good".

(b) The product of two odd numbers is odd, and the product of a multiple of 3 with anything is a multiple of 3. So the product of two "good" numbers is an odd multiple of 3, so is a "good" number.

Alternatively, we could write the two "good" numbers as $3(2m + 1)$ and $3(2n + 1)$ and multiply these together to get

$$3(2m+1).3(2n+1) = 9(4mn+2m+2n+1) = 9(2(2mn+m+n)+1) = 3(2(6mn+3m+3n+1)+1)$$

which is again a "good" number.

(c) If the product is "good", then it is odd and so the two integer factors are both odd. Furthermore the product is a multiple of 3, so at least one of the integer factors must be a multiple of 3, and therefore "good".

5. Road surfacing gangs are made up of a fixed number of workers and machines. Three gangs have surfaced 20 km of road in 10 days. How many additional gangs should be brought in if all the work is to be finished in another 15 days and there is still 50 km of road to be surfaced?

You should assume that their work does not interfere with each other.

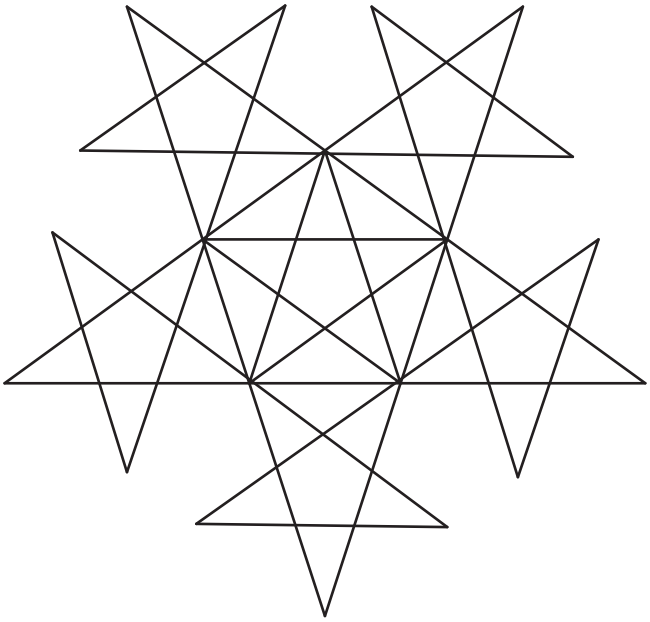
ANSWER 2 gangs

SOLUTION

There are a few different ways to organise the calculations for this question; this is just one such possibility.

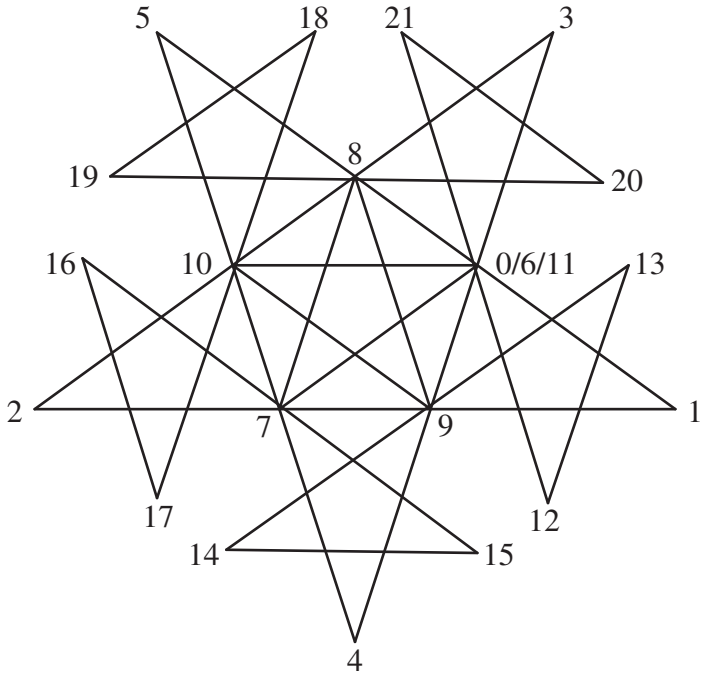
Three gangs surface 2 km of road in 1 day, so one gang can surface $\frac{2}{3}$ km of road in 1 day. There are 15 days remaining to surface 50 km of road, which requires $\frac{50}{15} = \frac{10}{3}$ km each day. This requires 5 gangs, so we need another 2 gangs.

6. This figure is to be drawn without lifting the pencil from the paper and without drawing any line twice. What is the smallest number of straight line strokes needed to draw the figure? (It is permitted for the pen to pass through individual points more than once. Also, all of the lines which appear to be straight in this figure really are straight.)



ANSWER 22

SOLUTION



A possible minimal route is shown starting at and ending at the point labelled 0. Every vertex has even degree, that is, at each vertex, there are an even number of edges meeting at the vertex. (*Vertex* or *node* is the term used to describe the meeting point of *edges* in a *graph* or network.) Because of this, the starting point could be chosen at any vertex on the diagram.

To minimise the number of straight lines drawn requires more thought: note that the answer stated is more than the number of straight lines in the figure.

There are 20 straight lines in the figure. If each of the five longest lines (1 to 2, 2 to 3 and so on) are drawn in a single stroke, then it will be impossible to draw the innermost star (7 to 8, 8 to 9 and so on), as each of the vertices 1, 2, 3, 4 and 5 only meets two straight lines. So one of these strokes must be broken into at least two parts to move onto a different line.

Another similar loop is 12 to 13 to 14 to 15 to 16 and so on, back to 12. If each of these edges is drawn in a single stroke, it will again be impossible to go from there to draw any of the other lines, for the same reason. So at least one of these strokes must also be broken.

There are therefore at least two broken straight lines, giving at least 22 strokes. We have demonstrated that it is possible to do this in 22 strokes, so this is the solution.

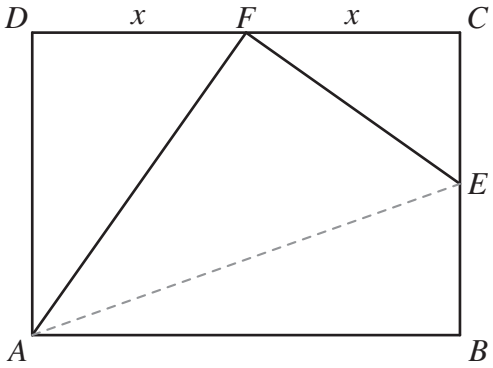
> Some follow-on questions would include asking a student to draw an accurate replica of this diagram, or to work out all of the angles involved (assuming that it is based on regular pentagons).

7. In the rectangle $ABCD$, let E be the midpoint of the side BC and F the midpoint of the side CD. If $\angle AFE = 90°$ and $BC = 2$, find CF.

ANSWER $CF = \sqrt{2}$

SOLUTION

We present two different approachs, and in each case, we take $CF = DF = x$. We begin by sketching the diagram described.



Approach 1

We note that since $\angle AFE = 90°$, then $\angle DFA + \angle CFE = 90°$, using the angle sum on a straight line.

But from the angle sum of a triangle, $\angle DFA + \angle DAF = 90°$ and $\angle CEF + \angle CFE = 90°$.

Hence $\angle DFA = \angle CEF$ and $\angle DAF = \angle CFE$. Therefore $\triangle DFA$ is similar to $\triangle CEF$.

This means that $AD : DF = FC : CE$. Using our known and unknown side lengths, this gives $2 : x = x : 1$, or as fractions, $\frac{2}{x} = \frac{x}{1}$. Hence $x^2 = 2$ and $x = \sqrt{2}$.

Approach 1

Applying Pythagoras's Theorem to the outer three right-angled triangles, we have:

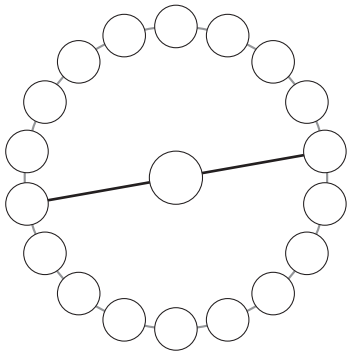
$$AF^2 = 2^2 + x^2, \quad EF^2 = 1^2 + x^2 \quad \text{and} \quad AE^2 = (2x)^2 + 1^2.$$

But $\triangle AFE$ is right-angled so $AF^2 + FE^2 = AE^2$. This leads to:

$$2^2 + x^2 + 1^2 + x^2 = (2x)^2 + 1^2$$

Noting that $(2x)^2 = 4x^2$, this simplifies to $4 = 2x^2$, giving $x = \sqrt{2}$ as before.

8. A circle has 18 cells round its perimeter and 1 cell in the centre to be filled with the numbers 1 to 19. Along any diameter, as illustrated in the diagram, the numbers in the three cells must add to a common total (that is, the sum of three diagonal cells is the same for every diagonal, similarly to a magic square). What are the possible values for the common total?



ANSWER 22, 30, 38

SOLUTION

Our strategy is as follows. We observe that the central cell appears in every diagonal, and so the sum of every pair of diagonally opposite cells must be equal. We start by determining which of the 19 numbers could have been left out for this to be possible.

We note that there are 9 pair of opposite cells, so the sum of all 18 outer cells must be a multiple of 9. The 19 cells sum to $1 + 2 + 3 + \cdots + 17 + 18 + 19$. This can also be written as $19 + 18 + 17 + \cdots + 3 + 2 + 1$ and, added together term by term, these sum to:

$$(1 + 19) + (2 + 18) + (3 + 17) + \cdots + (17 + 3) + (18 + 2) + (19 + 1) = 20 \times 19 = 380$$

$$\text{so that} \quad 1 + 2 + 3 + \cdots + 17 + 18 + 19 = \tfrac{1}{2} \times 380 = 190.$$

Now 189 is a multiple of 9 ($189 = 21 \times 9$), so we could have left out 1 to get a multiple of 9. Similarly, $189 - 9 = 180 = 20 \times 9$ is obtained by leaving out 10, and $180 - 9 = 171 = 19 \times 9$ is obtained by leaving out 19.

So the only plausible common totals are $21 + 1 = 22$, $20 + 10 = 30$ and $19 + 19 = 38$.

We now need to check that these totals are possible. We can line up the smallest number opposite the largest, the next smallest opposite the next largest, and so on, and this will work for all three cases: for the first, $2 + 19 = 21$, $3 + 18 = 21$, ..., $10 + 11 = 21$; for the second, $1 + 19 = 20$, $2 + 18 = 20$, ..., $9 + 11 = 20$, and for the third, $1 + 18 = 19$, $2 + 17 = 19$, ..., $9 + 10 = 19$.

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Mentoring Scheme

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Pythagoras

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Pythagoras of Samos (c. 569–475 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Pythagoras.html for more information.

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Version 1.1, Feb 2020

1. Near the beginning of the century we reached a time that could be represented as 01/01/01/01/01/01, in the form year/month/day/hour/minute/second. (We have assumed that the two-digit year number '01' means 2001 and that the hours use the 24 hour clock.) More recently, we passed a time represented in the same way as 12/12/12/12/12/12. How many seconds were there between these two times?

ANSWER 377 003 471 seconds

SOLUTION

There are 365 days in each year except for the leap years 2004, 2008 and 2012, so there are $12 \times 365 + 3 = 4383$ complete days between 1st January 2001 and 1st January 2013. 12th December 2012 is 20 days before 1st January 2013. Thus 4363 days have passed between the dates stated in the question.

This now makes $4363 \times 24 + 11 = 104\,723$ complete hours. The number of complete minutes is then $104\,723 \times 60 + 11 = 6\,283\,391$. We can then calculate the number of seconds as $6\,283\,391 \times 60 + 11 = 377\,003\,471$.

2. How many positive divisors does 6! have, including 1 and 6! ?

Recall from sheet 4, question 1 that $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$.

ANSWER 30

SOLUTION

We find the prime factorisation of 6!; we have

$$\begin{aligned} 6! &= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \\ &= 2 \times 3 \times 2^2 \times 5 \times 2 \times 3 \\ &= 2^4 \times 3^2 \times 5. \end{aligned}$$

Each of the powers of the prime factors can be considered independently, and so any factor of 6! must be a product of at most 4 2's, at most 2 3's and at most 1 5. There are 5 possibilities for the number of 2's: 0, 1, 2, 3 or 4; there are 3 possibilities for the number of 3's and 2 possibilities for the number of 5's. There are thus $5 \times 3 \times 2 = 30$ factors. These include 1 (which has 0 factors of 2, 0 factors of 3 and 0 factors of 5) and 6! itself (which has 4 factors of 2, and so on).

3. In this question, you should assume:

(a) the flesh of a fresh peach has a water content of 75% by weight;

(b) when left in the sun to dry, the flesh loses 80% of its water content;

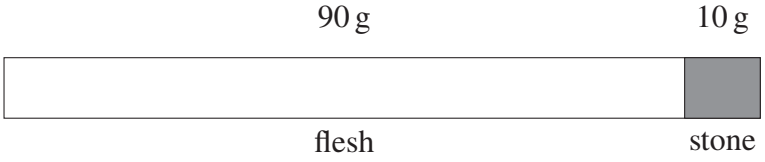
(c) the stone, removed before the drying process, accounts for 10% of the weight of a fresh peach.

How many grams of fresh peaches are needed to make up a 180 g pack of dried peaches with their stones removed?

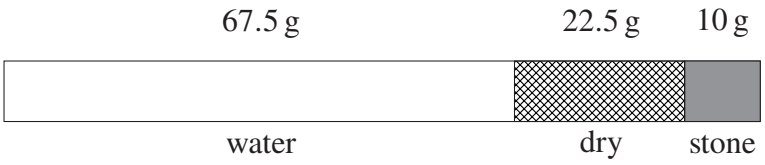
ANSWER 500 g

SOLUTION

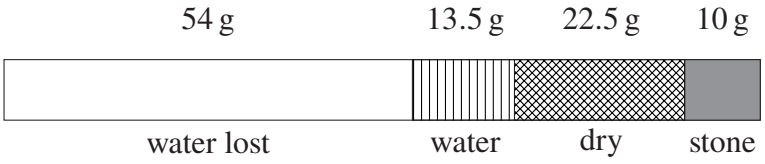
We can visualise this problem using bar modelling. Suppose we start with 100 g of peaches.



Next, we are told that the flesh is 75% water by weight. 75% of 90 g is 67.5 g:



80% of the water content is lost in drying, and 80% of 67.5 g is $0.8 \times 67.5\text{ g} = 54\text{ g}$:



So from the original 100 g, only 13.5 g + 22.5 g = 36 g is left after destoning and drying.

To produce 180 g of dried peaches, we therefore need $\dfrac{180\text{ g}}{36\text{ g}} \times 100\text{ g} = 500\text{ g}$ of raw peaches.

Another way of thinking about this final calculation is that the original peaches stoned and dried weigh only 36% of their original weight. Therefore, w g of original peaches produces $0.36w$ g of dried peaches. So to obtain 180 g of dried peaches, we need to solve $0.36w = 180$, giving $w = 500$.

> If you got the answer 600 g, it is probably because you took 75% of the weight of the entire fruit instead of 75% of the flesh.

4. Can you draw 8 straight lines (taken to extend infinitely in both directions) so that no two of the lines have the same number of intersection points with the other lines?

For example, if one line intersects with 6 other lines, then no other line may intersect with 6 lines.

ANSWER No, it is impossible

SOLUTION

For each of the eight lines to have a different number of intersections, the lines must cover all the possibilities of 0, 1, 2, 3, 4, 5, 6 and 7 intersections. For this to happen, the line with 7 intersections must intersect the line with 0 intersections, which is clearly impossible.

Alternatively, if a line has no intersections, then it must be parallel to all of the other lines. The only way that this can happen is if all 8 lines are parallel to each other, which would mean that every line has zero intersections. So every line must have at least one intersection, and hence they cannot each have a different number of intersections.

By either argument, the number '8' can be regarded as arbitrary, and can be replaced by n, provided that n is finite and at least 2.

The first of the two arguments works even if the lines are in 3-dimensional space rather than on a plane, whereas the second argument depends on the special properties of 2-dimensional space.

5. You are given a sequence of thirteen terms, each an integer, such that the fourth term is 17 and the eighth term is 20. You are also given that the sum of *every* three consecutive terms in the sequence is the same and that the sum of all thirteen terms is 217. What is the sequence?

ANSWER 17, 20, 13, 17, 20, 13, 17, 20, 13, 17, 20, 13, 17

SOLUTION

Let the terms of the sequence be $a_1, a_2, a_3, \ldots, a_{13}$. This is a convenient way to label the terms of a sequence.

The sum of every three consecutive terms is the same, so

$$a_1 + a_2 + a_3 = a_2 + a_3 + a_4.$$

Subtracting $a_2 + a_3$ from both sides shows that $a_1 = a_4$.

Likewise,

$$a_2 + a_3 + a_4 = a_3 + a_4 + a_5,$$

so $a_2 = a_5$.

Similarly, $a_3 = a_6$, $a_4 = a_7$, and so on; the sequence therefore repeats every three terms.

The fourth term is 17 and the eighth term is 20, so the sequence is 17, 20, x, 17, 20, x, 17, 20, x, 17, 20, x, 17, where x is yet to be determined.

The sum of these is $4(17 + 20 + x) + 17 = 165 + 4x = 217$, giving $x = 13$.

6. My calculator is broken so that the '0' key does not work and the display cannot display '0'. For example, I could not enter the calculation 10×3. Furthermore, the result of adding 37 and 13 is displayed as 5 (instead of 50), and 3 multiplied by 67 shows 21 (instead of 201). I multiply a one-digit number by a two-digit number on this calculator and the answer shown is 15. Which pairs of numbers could I have chosen which would give this result?

ANSWER $(1, 15)$, $(3, 35)$, $(5, 21)$, $(7, 15)$, $(2, 75)$, $(6, 25)$

SOLUTION

The answer to the multiplication must be either a two-digit or three-digit number. The possible answers are therefore 15, 105 and 150, and so we need to find products which give these. The pairs that work are:

$$\begin{aligned} 15 &= 1 \times 15 \\ 105 &= 3 \times 35 = 5 \times 21 = 7 \times 15 \\ 150 &= 2 \times 75 = 6 \times 25 \end{aligned}$$

Note that we cannot use $150 = 5 \times 30$ as we cannot type 30.

7. Zebedee is training on his bicycle. He returns by the same route that he used when cycling on the outward leg of his journey. He averages u kph on the outward leg and v kph on the return leg of his journey. Find Zebedee's average speed for his whole journey in terms of just u and v.

ANSWER $\dfrac{2}{\frac{1}{u}+\frac{1}{v}} = \dfrac{2uv}{u+v}$

SOLUTION

First let the distance travelled on each leg be d. The times on the outward and returns journeys are then

$$\frac{d}{u} \quad \text{and} \quad \frac{d}{v}$$

respectively.

Now we have the total time and the total distance, which is $2d$. Therefore the average speed is total distance divided by total time:

$$\frac{2d}{\frac{d}{u}+\frac{d}{v}}$$

You can simplify this to

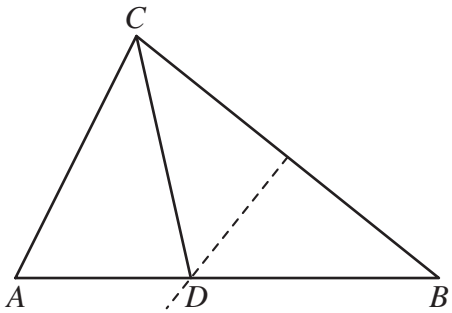
$$\frac{2}{\frac{1}{u}+\frac{1}{v}} = \frac{2uv}{u+v}.$$

This is known as the *harmonic mean* of u and v.

Notice that the distance does not appear in the final formula as it cancels.

8. Triangle ABC is scalene, that is, all three sides have different lengths. In addition, all three angles are acute. The perpendicular bisector of BC meets BA at the point D (where BA may be extended past A if necessary).

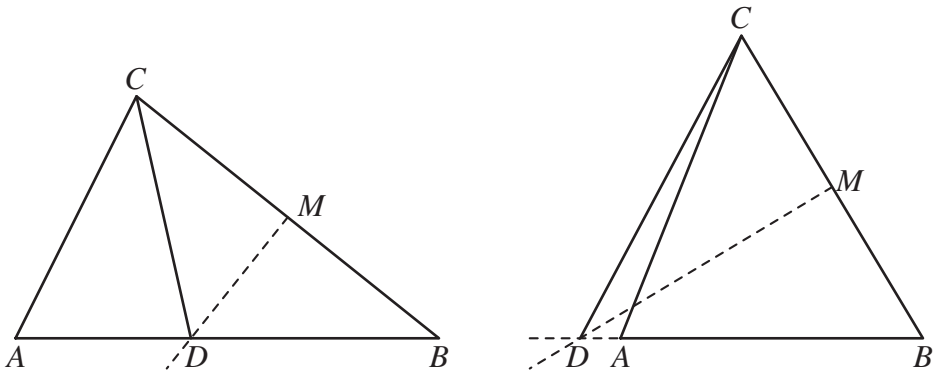
Prove that in the figure shown $\angle ADC = 2 \times \angle ABC$.



Prove that for some such triangles $\triangle ABC$, we instead have $\angle ADC + 2 \times \angle ABC = 180°$.

SOLUTION

Let M be the midpoint of BC. There are two cases to consider as shown by the two figures. On the left $AB > AC$, while on the right $AB < AC$. Note that on the left, D lies between A and B, while on the right, D lies beyond A. (The third possibility, $AB = AC$, cannot happen because the triangle is scalene.)



We first show that in both cases $\triangle BDM$ is congruent to $\triangle CDM$. DM is common to both triangles, $BM = CM$ since M is the midpoint of BC, and $\angle BMD = \angle CMD = 90°$ since MD is the perpendicular bisector of BC. Hence by the side–angle–side (SAS) rule, these two triangles are congruent.

It follows that $\angle DCM = \angle DBM$, and so $\triangle BCD$ is isosceles, and also that $\angle BDC = 180° - 2 \times \angle CBD$. Note that in both cases, $\angle CBD = \angle ABC$, as these describe the same angle, so $\angle BDC = 180° - 2 \times \angle ABC$.

In the left-hand figure, we then have $\angle ADC = 180° - \angle BDC = 2 \times \angle ABC$, where we have used the sum of angles on a straight line. (Note that we can shorten this part of the proof slightly if we use the result that the exterior angle of a triangle is equal to the sum of the interior opposite angles.)

In the right-hand figure, we instead have $\angle ADC = \angle BDC = 180° - 2 \times \angle ABC$, so $\angle ADC + 2 \times \angle ABC = 180°$.

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Pythagoras

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Pythagoras of Samos (c. 569–475 BCE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Pythagoras.html for more information.

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Version 1.1, Mar 2020

1. At most how many Mondays can there be in 45 consecutive days?

ANSWER 7

SOLUTION

In 42 consecutive days, there are always 6 Mondays; there would be 7 Mondays in 43 to 48 consecutive days if the first day is a Monday but only 6 Mondays if the first day is a Tuesday; in 49 consecutive days there are always 7 Mondays. So 45 consecutive days always contain 6 or 7 Mondays.

2. How many digits are there in the smallest number composed entirely of fives (for example, 5555555) that is divisible by 99?

ANSWER 18

SOLUTION

To be divisible by 9, the digits must add to a multiple of 9. As the digits are all the same, the answer must have an even number of digits to be divisible by 11 (as 55, 5500, 550000 and so on are all divisible by 11).

3. Calculate the sum $1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + \cdots + 34 + 34 + 35 + 35 + 36$.

ANSWER 1295

SOLUTION

We could split this sum into two parts with an equal number of terms in each, namely:

$$1 + 2 + 3 + 4 + \cdots + 35 +$$
$$2 + 3 + 4 + 5 + \cdots + 36$$

These could be represented by two triangular dot patterns. We can slot these triangles together by reflecting one of the patterns. Putting them together gives a 35×37 rectangle of dots.

Alternatively note that there are 35 numbers in each line above. We can rewrite the second line in reverse to give

$$1 + \ 2 + \ 3 + \ 4 + \cdots + 35 +$$
$$36 + 35 + 34 + 33 + \cdots + \ 2$$

and we note that each column sums to 37, so the total is $35 \times 37 = 1295$.

4. You have two jugs with capacities of exactly 7 litres and 5 litres, and a tap for a supply of water. Using only these jugs, you have to pour into a bottle (capacity about $1\frac{1}{2}$ litres) with exactly 1 litre of water.

A jug may be filled to the top from the tap or from the other jug, a jug may be entirely poured into the other jug if there is sufficient space available, or a jug may be completely emptied.

How do you pour exactly 1 litre of water into the bottle?

SOLUTION

The most efficient solution (requiring the least number of pourings) is the following, though there are infinitely many other possibilities.

	7 litre jug	5 litre jug
From tap to 5 litre jug	0	5
From 5 litre jug to 7 litre jug	5	0
From tap to 5 litre jug	5	5
From 5 litre jug to 7 litre jug	7	3
Empty 7 litre jug	0	3
From 5 litre jug to 7 litre jug	3	0
From tap to 5 litre jug	3	5
From 5 litre jug to 7 litre jug	7	1
From 5 litre jug to bottle		

The 5 litre jug is filled three times in this solution. If we wanted exactly 1 litre left over in total, we would empty the 7 litre jug at the end, so the 7 litre jug is emptied twice. Note that this leads to the equation $3 \times 5 - 2 \times 7 = 1$; any other solution must result in a similar equation.

5. Matthew sends an encrypted message to Eman by using an agreed key word which has no letter repeated, e.g. XIHLBW. He replaces an original 'a' with X, 'b' with I, 'c' with H, 'd' with L, 'e' with B and 'f' with W, then he continues through the remaining letters of the alphabet. He replaces 'g' with Y (which is the next unused letter after W), 'h' with Z, and then he continues from the start of the alphabet so that 'i' is replaced with A, 'j' with C and so on until the substitution cypher is complete.

What letter does he replace with R?

Matthew and Eman challenge Hayley to decrypt their messages. Hayley knows their system and that they are using a six letter key word. How may possible such keywords are there?

ANSWER 'v'; 165 765 600

SOLUTION

The complete substitution is as follows:

original	a b c d e f g h i j k l m n o p q r s t u v w x y z
replacement	X I H L B W Y Z A C D E F G J K M N O P Q R S T U V

Matthew has 26 choices for the first letter of the key word, 25 for the second and so on.

That makes $26 \times 25 \times 24 \times 23 \times 22 \times 21 = 165\,765\,600$ possible keys.

In practice, by finding the letter frequencies in the encrypted message, Hayley may find that B, say, is the most frequent letter in the cipher and suspect that it represents 'e'. Continuing with this method, she might well decrypt the whole message or at make sense of part of it. Also, since the cypher is mostly some overlapping Caesar shift cyphers, trying different shifts is likely to reveal a good portion of the message without excessive effort. Both of these approaches are far more efficient and effective than trying every single possible key.

In many books and papers on encryption, the person sending the message is called Alice, the person intended to receive it is Bob and the eavesdropper is Eve. Other characters also appear.

6. Mandy tries a card 'trick' on her friends. Out of sight, she arranges eight cards numbered 1 to 8 face down in a specific order. She hands the pack face-down to one of her friends and gives them these instructions:

(a) Turn over the top card and lay it face up on the table.

(b) Transfer the (new) top card to the bottom of the pack.

(c) Repeat (a) and (b) in turn until all the cards have been laid out on the table.

What original card order did she use to make the cards come out in order from 1 to 8?

Can you use the method you have found to do this for cards numbered 1 to 16?

ANSWER 1 5 2 7 3 6 4 8; 1 9 2 13 3 10 4 15 5 11 6 14 7 12 8 16

SOLUTION

With the cards face-down, the top card must be 1, then the third card must be 2, and so on, giving the order $1x2x3x4x$. After the first four cards are laid down, the remaining four cards are those indicated by x. The remaining four cards must give 5, 6, 7, 8 when laid down. Therefore they must be in the order $5y6y$ to give the 5 and 6, and finally the two ys must be 7 and 8 in that order, giving the complete order

$$1\ 5\ 2\ 7\ 3\ 6\ 4\ 8$$

For 16 cards, we can argue similarly. The original order must be

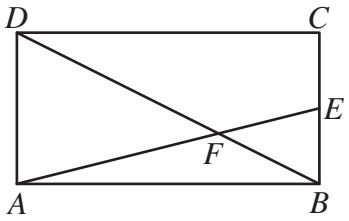
$$1\ x\ 2\ x\ 3\ x\ 4\ x\ 5\ x\ 6\ x\ 7\ x\ 8\ x$$

where the xs are cards numbered between 9 and 16. Once the first eight cards are laid down, the remaining cards should be ordered in the same way as the first answer, but renumbered by adding 8 onto them, giving the final order:

$$1\ 9\ 2\ 13\ 3\ 10\ 4\ 15\ 5\ 11\ 6\ 14\ 7\ 12\ 8\ 16$$

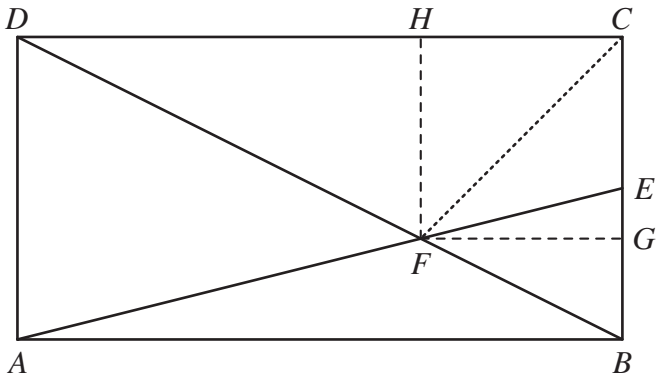
You might like to consider how the answer would change if the number of cards were not a power of 2.

7. In the rectangle $ABCD$, $AB = 4$ and $BC = 2$. Let E be the midpoint of the side BC, and let AE cut BD at F. Prove that $\angle BCF = 45°$.



SOLUTION

If we draw in the line segment CF and draw perpendiculars FG and FH to the sides BC and CD of the rectangle, we see that $\angle BCF = 45°$ if these two perpendiculars have equal lengths.



We show that this is the case by firstly noting that $\triangle DFA$ is similar to $\triangle BFE$. This is because $\angle EBF = \angle ADF$ and $\angle BEF = \angle DAF$ as BC is parallel to DA. Since $BE : DA = 1 : 2$, it follows that $BF : DF = 1 : 2$. Hence F is a point of trisection on BD (that is to say, $BF = \frac{1}{3}BD$).

Using similar triangles based on $\triangle ABD$, the distance of F from BC equals $\frac{1}{3} \times 4 = \frac{4}{3}$ (or $1\frac{1}{3}$).

Similarly, the distance of F from AB is $\frac{2}{3} \times 1 = \frac{2}{3}$, so the distance of F from CD equals $1 - \frac{2}{3} = \frac{4}{3}$.

Hence $\angle BCF = 90°$.

An alternative approach is to use $BF : DF = 1 : 2$ to prove that CF extended meets AB at its midpoint, but we do not give the details of this argument here.

8. In the summer term, Sal walks to school starting soon after 7 am when the hour hand is at an angle of 110° to the minute hand. She arrives at school before 8 am when the hands are again at an angle of 110° to each other. How many minutes does she take to walk to school?

ANSWER 40 minutes

SOLUTION

The most efficient way to answer this question is to note that the minute hand passes the hour hand and that it rotates 220° more than the hour hand. Now in one hour, the minute hand rotates 330° more than the hour hand, because the minute hand rotates by 360° while the hour hand rotates only 30°.

Therefore 220° takes exactly $\frac{2}{3}$ hour, and this is the time that Sal takes to walk to school.

An alternative, though longer, approach is to find out when Sal leaves the house and when she arrives at school.

Let t be the number of minutes after 7 am when she leaves. The minute hand has turned through $6t°$ from the 'twelve o'clock' position. The hour hand has turned through $\frac{t}{2}°$ since 7 am but is already 210° ahead of the 'twelve o'clock' position. Now we can form an equation:

$$\begin{aligned}(210 + \frac{t}{2}) - 6t &= 110\\ 420 + t - 12t &= 220\\ 200 &= 11t\end{aligned}$$

We could now work out the exact time of departure, which is $7.18\frac{2}{11}$ am, but this turns out not to be necessary.

Now let T be the number of minutes before 8 am that she arrives at school. By a similar argument, this gives:

$$\begin{aligned}(120 + \frac{T}{2}) - 6T &= 110\\ 240 + T - 12T &= 220\\ 11T &= 20\end{aligned}$$

As before, we can calculate the arrival time as $7.58\frac{2}{11}$ am. Simpler, though, is to note that the number of minutes between 7 am and 8 am when she was *not* walking is $t + T$; we have $11(t + T) = 200 + 20 = 220$, so $t + T = 20$, hence she was walking for the other 40 minutes of the hour.

A slight variant on this is to let T instead be the number of minutes after 7 am that she arrives at school. In this case, the equation is $6T - (210 + \frac{T}{2}) = 110$ giving $11T = 640$. We can now calculate $11(T - t) = 440$, so $T - t = 40$ as before.

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Pythagoras

Sheet 7

Solutions and comments

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1. A large rectangle field has area 55 296 m^2. One side is $1\frac{1}{2}$ times an adjacent side. What is the perimeter of the field?

ANSWER 960 m

SOLUTION

An algebraic way to proceed would be to let h be the shorter side. Then the longer side is $1.5h$. We then have the equation:

$$\begin{aligned} 1.5h^2 &= 55296 \\ h^2 &= 36864 \\ h &= 192 \\ perimeter &= 2 \times h + 2 \times 1.5h \end{aligned}$$

2. Yesterday Lucian Schwindler, the dodgy money changer, exchanged 2 rotts for 3 weilers. Today he exchanged 2 weilers for 1 rott and 3 hunds at the same rate. How many hunds are there in a weiler?

ANSWER 6

SOLUTION

The first statement tells us that 2 weilers is equivalent to $\frac{4}{3}$ rott so that the second statement tells us that 3 hunds is equivalent to $\frac{1}{3}$ rott. Hence there are 9 hunds to a rott. Now we have

$$18 \text{ hunds} = 2 \text{ rotts} = 3 \text{ weilers}.$$

3. A runner covers 5 km in 16 minutes 40 seconds. Along the course, feeling confident that she is fit, the runner increases speed so that each kilometre is run in a time 5 seconds shorter than the previous kilometre. How long does she take over the last kilometre?

ANSWER 3 minutes 10 seconds

SOLUTION

This is another question where it is possible to set up an algebraic solution. Suppose the time of the last kilometre is T seconds. Then we can form the equation

$$\begin{aligned} (T + 20) + (T + 15) + (T + 10) + (T + 5) + T &= 16 \times 60 + 40 \\ 5T + 50 &= 1000 \\ 5T &= 950 \\ T &= 190 \end{aligned}$$

Another approach is to note that the last kilometre takes 10 seconds less than the average time for each kilometre which is $T + 10$. The average time is 16 minutes 40 seconds $\div 5$ = 3 minutes 20 seconds so the last lap takes 3 minutes 10 seconds.

4. At a party, some boys and girls paired up to dance. It turns out that $\frac{3}{5}$ of the boys are paired up with girls while $\frac{2}{3}$ of the girls are paired up with boys. What fraction of the children are paired up boy with girl?

ANSWER $\frac{12}{19}$

SOLUTION

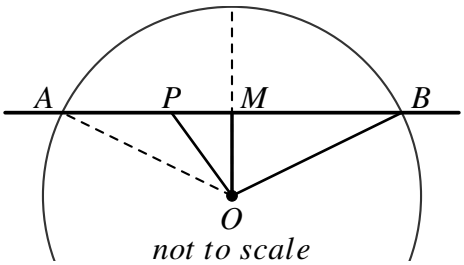
Suppose there are p pairs of boys and girls. Then the number of girls present must be $\frac{3}{2}p$. Similarly the number of boys present is $\frac{5}{3}p$. Thus there are $(\frac{3}{2}+\frac{5}{3})p = \frac{19}{6}p$ children at the party altogether while $2p$ of them are paired up. The fraction required is:

$$\frac{2}{\frac{19}{6}} = 2 \times \frac{6}{19} = \frac{12}{19}$$

5. The diameter of a circle with centre O is 110 cm. A point P lies on a chord AB of the circle so that $AP = 30$ cm and $BP = 60$ cm. Find the length of OP.

ANSWER 35 cm

SOLUTION



Let M be the midpoint of AB. If we join O to A, P, M and B we can see three right angled triangles. (We only need two of them.) $OA = OB = 55$ cm and $AM = BM = 45$ cm. *We will omit units from the equations*. Applying Pythagoras' Theorem to $\triangle OMB$ (or $\triangle OMA$) we obtain

$$OM^2 + 45^2 = 55^2$$
$$\text{from which} \quad OM^2 = 1000$$

At this point you may think you need to do something like $OM = 31.6227766\ldots$ but it is wise to wait. The next stage is to apply Pythagoras' Theorem to $\triangle OMP$. We need $MP = 15$ cm.

$$OM^2 + 15^2 = OP^2$$

In other words we can replace OM^2 with 1000 and carry on with exact numbers. Hence

$$OP^2 = 1225$$
$$OP = 35$$

Doing this without a calculator requires some recognition of higher square numbers. It is worth noting that all the figures given can be reduced by a factor of 5. Taking the diameter as 22, $AP = 6$ and $BP = 12$ simplifies the arithmetic.

There is a nice way of working out "half squares". Study the following:

$(\frac{1}{2})^2 = \frac{1}{4}$, $(1\frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4} = 2\frac{1}{4}$, $(2\frac{1}{2})^2 = 6\frac{1}{4}$, $(3\frac{1}{2})^2 = 12\frac{1}{4}$, $(4\frac{1}{2})^2 = 20\frac{1}{4}$ and so on

. . . scaled up . . .

$5^2 = 25$, $15^2 = 225$, $25^2 = 625$, $35^2 = 1225$, $45^2 = 2025$, $55^2 = 3025$, . . .

Closer study should reveal triangle numbers embedded in the results.

6. Show that the tens digit of every power of 3 is even.

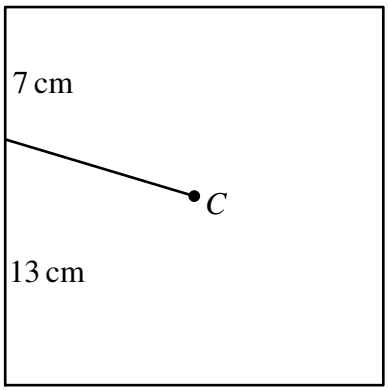
SOLUTION

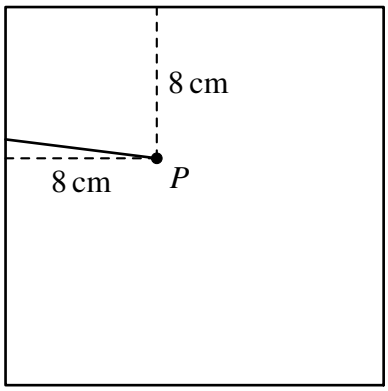
We don't really need to be concerned with the hundreds, thousands and higher order digits of the powers of 3. In fact, the question does not even ask us about the units digit though it can be shown that if you multiply 1, 3, 7 or 9 by 3, then you will get one of these digits again. This suggests one way of tackling the problem. Write all the powers of 3 keeping only the tens and units digits. They come to $3, 9, 27, (81, 43, 29, 87, 61,) 81, 43, \ldots$. We obtain a cycle of five numbers picked out by brackets.

Here is a more algebraic approach.

Suppose we have a power of 3 which can be written as $100x + 20t + u$ where t and u are single digits but x can be any integer. In other words, we have got fed up of writing powers of 3 but notice that the rule seems to be true for the last number we have written down. Now work out $3(100x + 20t + u) = 300x + 60t + 3u$ to find the next power of 3. Since we have shown that $3u$ has a value with an even tens digit when $u = 1, 3, 7$ or 9, we can deduce that the next number we write down will also obey the rule.

7. Ravina has a square (flat topped) birthday cake with edges of 20 cm. She wishes to cut it with straight vertical cuts into five equal portions by volume to share with her four friends.
(a) Suppose she makes a cut from the centre C to the edge as shown in the left hand figure. Where must she make the other four cuts if they are also made from C?
(b) Suppose the first cut starts at P is as shown in the right hand diagram but goes to the same point on the edge as in case (a). Where must she make the other four cuts if they all start at P?
(c) Marzipan is placed all the way round the side and top of the cake. Ravina uses one of the methods in case (a) or (b) and gets more marzipan than any of her four friends. Which slice of cake does she take?

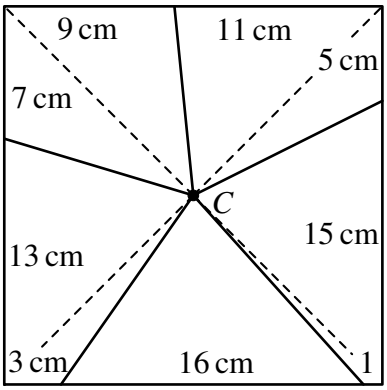


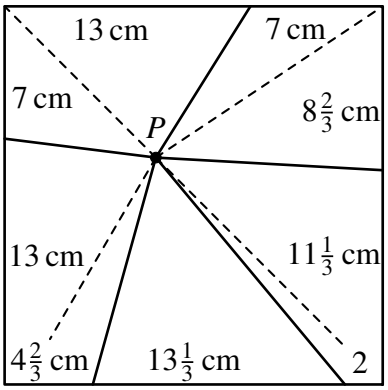


Solution

Solution: **(c) Ravina chooses the top left slice in case (b)**

(a)





(b)

Case (a). The height of all the triangles shown in the figure is 10 cm. Most slices are assembled from two triangles such as the top left slice. In this slice, the lower triangle has area $\frac{1}{2} \times 7 \times 10\,\text{cm}^2$. The upper triangle has area $\frac{1}{2} \times 9 \times 10\,\text{cm}^2$. This totals $80\,\text{cm}^2$, one-fifth of the area of the top of the cake. More usefully, it uses one-fifth of the perimeter of the cake so all we have to do is mark points every 16 cm round the edge.

Case (b) is more difficult because the height of each triangle varies round the figure but each slice must take 80 cm^2 of the area of the top of the cake. We will work round clockwise from the top left corner labelling the edges $a, b, c, \ldots g$ and ignore units in the calculations.

Slice (1) Area = $\frac{1}{2} \times 7 \times 8 + \frac{1}{2} \times a \times 8$ so $28 + 4a = 80$. This gives $a = 13$.
Slice (2). $b = 20 - 13 = 7$. Area = $\frac{1}{2} \times 7 \times 8 + \frac{1}{2} \times c \times 12$ so $28 + 6c = 80$.
This gives $3c = 26$ so $c = 8\frac{2}{3}$.
Slice (3). $d = 20 - 8\frac{2}{3} = 11\frac{1}{3}$ which does not provide sufficient base for this slice.
Area = $\frac{1}{2} \times 11\frac{1}{3} \times 12 + \frac{1}{2} \times e \times 12$ so $68 + 6e = 80$. Hence $e = 2$.
Slice (4). Area = $\frac{1}{2} \times f \times 12$ so $6f = 80$. Hence $f = 13\frac{1}{3}$ can be fitted along this side.
Slice (5). Check. $g = 20 - 13\frac{1}{3} - 2 = 4\frac{2}{3}$.
Area = $\frac{1}{2} \times 4\frac{2}{3} \times 12 + \frac{1}{2} \times 13 \times 8$ which comes to 80.

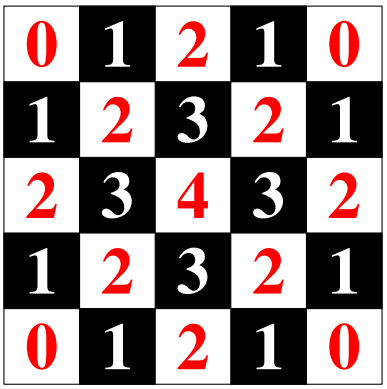
(c) We now see that the top left slice in case (b) will give more marzipan around the side than any of the other slices. The marzipan on top is equally distributed and does not affect the outcome.

8. A 5 by 5 grid of squares is set up and 25 counters provided, each red on one side and white on the other. The counters are placed on the grid one at a time. When a counter is put down on a particular square, any counters on neighbouring squares are turned over. Two squares are neighbours if they have a common edge; two squares sharing just one vertex do not count as neighbours. The aim is to finish with all the counters showing their red face.
In the process of placing the 25 counters on the board, many counters were turned over, some more than once. In total, how many times were counters turned over?

ANSWER 40

SOLUTION

Where is the best place to put down the first counter? Symmetry may suggest it be done at the centre. Working outwards we can see that the number of times a counter is turned over will be as shown in the diagram. For example, the centre square will be turned over 4 times because it has 4 neighbours on which counters will be placed. On the other hand, the counter to the left of it will only be turned over 3 times because the centre counter is one of its neighbours but the centre counter is already in position when counters round the left counter are put down.



We now have a rule about which way to place the counters. The centre counter and all those on 'white' squares must be placed red face up.

What will happen if we adopt a different method of putting down the counters such as starting at a corner? We will certainly have a different pattern of placing the counters red or white face up. But does it affect the final answer? No.

Consider the number of internal boundaries between squares. It comes to 40. Why can this insight yield the final answer without knowing in what order the counters were placed on the squares? Think about what causes a counter to be turned. It is when it is already in place and a new counter is put down next to it. It is as though the new counter issues an instruction to the old counter across the boundary between them to turn over. After that, no further instructions can cross that boundary. This does not stop a newer counter issuing another instruction to the old counter across a different boundary. Hence the number of such instructions given is the same as the number of internal boundaries.

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Pythagoras

Sheet 8

Solutions and comments

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Version 1.1, May 2020

1. If a and b are positive integers and $a + b < 10$, how many different values can ab take?

ANSWER 16

SOLUTION

Counting in a systematic way is the most reliable method. These are the multiplication tables, with each row worked out up to the point where the two numbers being multiplied sum to 9 (since $a + b < 10$). The first occurrence of each product is printed in bold (starting from the top row).

	1	2	3	4	5	6	7	8
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2	2	4	6	8	**10**	**12**	**14**	
3	3	6	**9**	12	**15**	**18**		
4	4	8	12	**16**	**20**			
5	5	10	15	20				
6	6	12	18					
7	7	14						
8	8							

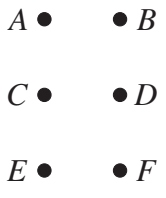
You can see that we have excluded all products below a diagonal line from top left to bottom right because $a \times b = b \times a$, and certain products such as 3×4 because $3 \times 4 = 2 \times 6$. This leaves those products to be counted picked out in bold. All other numbers in the table are repetitions. There are 16 bold numbers in total.

2. Six dots are arranged in three rows and two columns as shown. How many triangles have their vertices at three of the points?

Note that we may not choose three points that lie on a straight line.

ANSWER 18

SOLUTION



The most obvious method is to use a systematic count connecting every possible triple of three points not in a straight line. Labelling the points as shown, these triples are, in alphabetical order:

$$(A,B,C),\ (A,B,D),\ (A,B,E),\ (A,B,F),\ (A,C,D),\ (A,C,F),$$
$$(A,D,E),\ (A,D,F),\ (A,E,F),\ (B,C,D),\ (B,C,E),\ (B,C,F),$$
$$(B,D,E),\ (B,E,F),\ (C,D,E),\ (C,D,F),\ (C,E,F),\ (D,E,F)$$

> Using binomial coefficients, this question can be answered more efficiently by noting that there are $\binom{6}{3} = 20$ sets of vertices of which two sets, namely (A, C, E) and (B, D, F), lie on a straight line.

3. In a certain association, every member can vote for the president. The current president has been elected with twice as many votes as were received by the only other candidate. Knowing that 3 members did not vote and that the current president obtained the support of 64% of *all* the members, how many members are there?

ANSWER 75

SOLUTION

An arithmetical method is to write out a sequence of calculations:

	64% voted for president
so	32% voted for the other candidate
so	96% of members vote
so	4% of members did not vote.
Now	4% is 3 members
so	100% is $25 \times 3 = 75$ members

This calculation can also be represented using bar modelling.

Alternatively, we can work algebraically. Let n be the number of members, then the number of members who voted for the current president is $0.64n$ and the number who voted for the other candidate is $0.32n$. We can now set up the equation

$$0.64n + 0.32n + 3 = n$$
$$\text{so} \quad 0.04n = 3$$
$$\text{giving} \quad n = 3 \div 0.04 = 3 \times 25$$

4. Without using a calculator, place the numbers $3\sqrt{11}$, $4\sqrt{7}$, $5\sqrt{5}$, $6\sqrt{3}$ and $7\sqrt{2}$ in increasing order.

ANSWER $7\sqrt{2}, 3\sqrt{11}, 6\sqrt{3}, 4\sqrt{7}, 5\sqrt{5}$

SOLUTION

We note that if a and b are two non-negative real numbers, then $a < b$ if and only if $a^2 < b^2$. So by considering the squares of these numbers, we will be able to order them. (This is not the case if the numbers might be negative, for example $-3 < 2$ but $(-3)^2 > 2^2$.)

The squares are:

$$(3\sqrt{11})^2 = 9 \times 11 = 99$$
$$(4\sqrt{7})^2 = 16 \times 7 = 112$$
$$(5\sqrt{5})^2 = 25 \times 5 = 125$$
$$(6\sqrt{3})^2 = 36 \times 3 = 108$$
$$(7\sqrt{2})^2 = 49 \times 2 = 98$$

Therefore the original numbers are, in increasing order, $7\sqrt{2}$, $3\sqrt{11}$, $6\sqrt{3}$, $4\sqrt{7}$ and $5\sqrt{5}$.

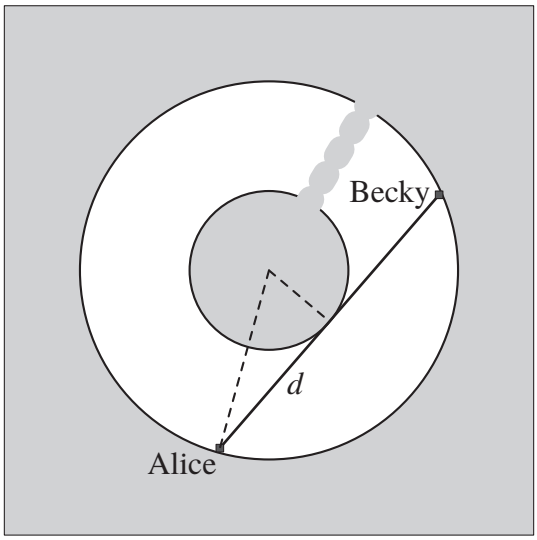
5. Alice and Becky are the only inhabitants of an atoll whose shape is bounded by two concentric circles with radii 1 km and 6 km. There is a lagoon of water inside the smaller circle accessed by a narrow channel from the sea. After their huts have been destroyed by a storm, they decide to rebuild the huts as far from each other as possible but located so that they can walk in a straight line from one to the other entirely on land (right up to the edge of the lagoon). How far apart are the huts?

'Concentric' means that the circles have the same centre.

ANSWER $2\sqrt{35}$ km

SOLUTION

The atoll (not to scale)



In the right angled triangle on the diagram, two sides are radii of the two circles. If the distance from Alice's hut to Becky's hut is $2d$, then

$$d^2 + 1^2 = 6^2$$

so $d^2 = 35$ and the distance between the huts is $2\sqrt{35}$ km.

6. Giovanni is a bit of a comedian and is programming his toy robot to move as though it is drunk so that he can show it off in his next robotics lesson at school and raise a laugh. He sets it to perform the following 7 actions in order over and over again:

- it takes 1 step forward
- it turns 90° to the right
- it takes 2 steps forward
- it turns 90° to the right
- it takes 1 step forward
- it turns 90° to the left
- it takes 1 step backwards

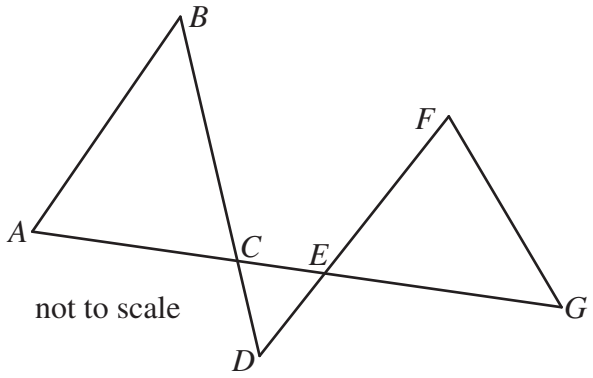
After 186 steps (not actions) the battery fails. Each step is 4.5 cm long. How far is the robot from its starting point when it stops?

ANSWER 9 cm

SOLUTION

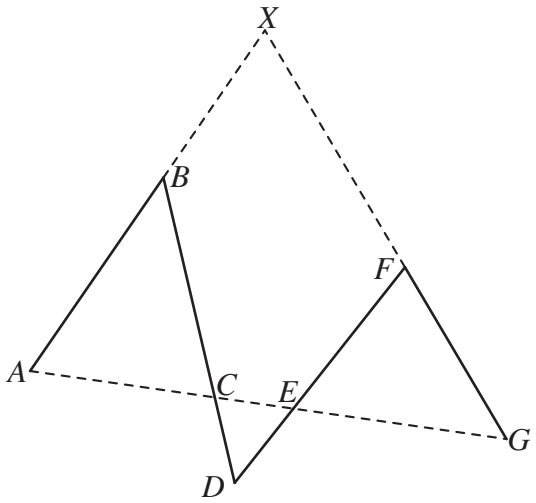
Assume the robot starts facing east. After the 7 actions described, it is 1 step south of its original position facing south having taken 5 steps. After doing the 7 actions three more times it is back where it started facing east and has taken a total of 20 steps. Thus after 180 steps it is also back where it started. It now only takes 6 more steps before the battery runs out, and ends up 2 steps south of where it started, which is a distance of 9 cm.

7. In the shape shown in the diagram, $\angle ABC = 70°$, $\angle EFG = 80°$ and $\angle CDE = 95°$. Calculate the acute angle at which AB meets GF when both are extended.



ANSWER 55°

SOLUTION



Let X be where AB extended meets GF extended, as shown.

One way to solve this question is to ignore the line $ACEG$ entirely. Consider the quadrilateral $BDFX$ whose angles add up to 360°.

$$\begin{aligned} \angle XBD &= 180° - 70° = 110° \quad \text{(angles on a straight line)} \\ \angle XFD &= 180° - 80° = 100° \quad \text{(angles on a straight line)} \\ \angle BDF &= 95° \quad \text{(given)} \end{aligned}$$

Hence $\angle BXF = 360° - 110° - 100° - 95° = 55°$.

As a second approach, since $\angle CDE = 95°$, it follows that $\angle DCE + \angle DEC = 85°$ (angles in a triangle). Therefore $\angle ACB + \angle FEG = 85°$, as these are vertically opposite angles to the previous two.

Now we are given $\angle ABC$ and $\angle EFG$, so we have

$$(\angle ABC + \angle ACB) + (\angle EFG + \angle FEG) = 70° + 80° + 85° = 235°.$$

Using the angle sum in the two triangles $\triangle ABC$ and $\triangle EFG$, we find that

$$\angle BAC + \angle EGF = 2 \times 180° - 235° = 125°$$

leaving $\angle AXG = 55°$ (from the angle sum in $\triangle AGX$).

A third approach is more algebraic. Let us suppose that $\angle BCA = x°$.

Then we can successively calculate

$$\begin{aligned} \angle XAG = \angle BAC &= 180° - 70° - x° = 110° - x° \\ \angle DCE = \angle BCA &= x° \\ \angle DEC &= 180° - 95° - x° = 85° - x° \\ \angle FEG = \angle DEC &= 85° - x° \\ \angle XGA = \angle FGE &= 180° - (85° - x°) - 80° = 15° + x° \end{aligned}$$

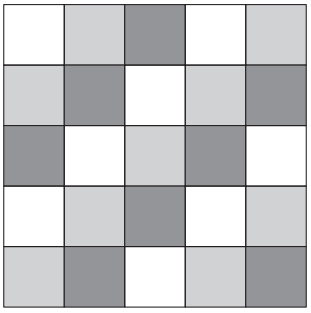
Hence $\angle XAG + \angle XGA = (110° - x°) + (15° + x) = 125°$. Then the angle sum of $\triangle XAG$ gives $\angle AXG = 55°$ as before.

8. A board consists of a 5 by 5 grid of unit squares. It is to be covered by eight rectangular shapes measuring 3 by 1 units in such a way that the rectangles do not overlap and that their edges follow the lines of the grid. One of the squares of the grid will be left uncovered. Find all the possible positions of this uncovered square.

ANSWER Only the centre square

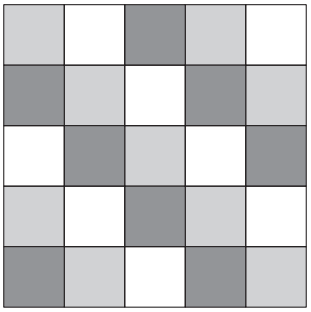
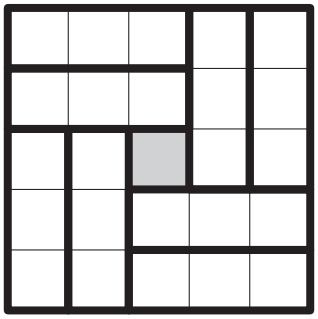
SOLUTION

One way to colour the grid is shown below.



Each rectangle covers one square of each colour. As there are 8 white and 8 dark squares but 9 grey squares, the uncovered square must be a light grey one.

We can also colour the diagonals in the opposite direction (top left to bottom right), by reflecting the given board. Again, we find that the uncovered square must be a light grey one.



The only square which is coloured light grey in both of these colourings is the centre square, so any arrangement of the rectangles must the leave the centre square uncovered. It is easy to show that there are two possible arrangements of the rectangles which leave this square uncovered; here is one, and the other is just its reflection: